Sequences;

Definition; let f;N→R be a function and let f(n)=a_n .Then a1,a2,a3....... Is called the sequences in R determined by the function f and is denoted by (a_n). (a_n) is called the nth term of the sequence

The range of the function f. which is subset of R is called the range of the sequences

Examples:

1.The function f;N→R given by f(n)=n determines the sequences 1,2,3,......

2. The function f;N→R given by f(n)=n^2determines the sequence 1,4,9.......

3. The function f;N→R given by f(n)=$(-1)^n$ determines the sequence -1,1.-1,1...... Thus the term of the sequence need not the distinct.The range of the sequence is (-1,1).Thus the we get therange of the sequence may be finite or infinite.

4. The sequence($(-1)^{n+1}$ is given by 1,-1,1,-1.... The range of this sequence is also (1,-1).However

weknow that the sequence $(-1)^n$ and$(-1)^{n+1}$ are different.The first sequence startswith -1 and the second sequence starts with 1.

5.The constant function f;N→R given by f(n)=1 determines the sequences 1,1,1,....Such the sequence is called is constant sequences.

6. The function f;N→R given by f(n)=$\frac{1}{2}$n if nis even

½(1-n) if n is odd

Determiningthe sequence 0,1,-1,2,-2.....n,-n.The range of the sequence is z.

7. The function f;N→R given by f(n n/n+1. Determines the sequence ½,2/3,3/4,.........n/n+1.

8. The function f;N→R given by f(n)=1/n determines the sequence 1,1/2,1/3,.........

9. The function f;N→R given by f(n)=2n+3 determines the sequence 5,7,9..........

10.let x€R .The function The function f;N→R given by f(n)=x^{n-1} determines the geometric sequences 1,x,x^2.......x^n.

11. The sequences(-n) is given by -1,-2,-3,......-n. The range of the this sequence is the set of the all negative integers.

12. A sequence can also be described by specifying the first few terms and starting a rule of determining an is a term of the previous term of the sequence. For example let a_1=1,a_2=1 and a_n=a_{n-1}+a_{n-2}. Thus obtain the 1,1,2,3,5,8......Thus the sequence is called FIbonacis sequence

13. Leta_1=$\sqrt{2}$ and a_{n+1}=$\sqrt{2+a_n}$. This defines the sequences $\sqrt{2}$,$\sqrt{2+\sqrt{2}}$

Bounded sequences;

Definition;A sequence(a_n) is said to be bounded above if there exists a real number k such that $a_n \leq k$ for all n € **N** .Then k is called upper bound of the sequences.

A sequences (a_n) is said to be bounded below ifthere exists a real number k such that $a_n \geq k$ for all n. The k is called a lower bounded of the sequence.

A sequence(a_n) is said to be bounded sequences if it is both bouned above and below.

Note;

A sequences (a_n) is bounded ifthere exists a real number k≥0 such that (a_n)≤k for all n

The least upper bound and greatest lower bound of sequences can be difined as in 1.4.

Example;

1.Consider the sequences 1,1/2,1/3.....1/n. Here l.u.b and 0 is the g.l.b .It is bounded sequences

2.The sequence 1,2,3,...n is bounded below and not bounded above .1 is the glb of the sequence

3.The sequence -1,-2,-3,.....-n is bounded above but not bounded below -1 is the lub of the sequences

4.1,-1,1,-1.... Is the bounded sequences 1 is the lub and -1 is the glb of the sequences

5.Any constant sequence is a bounded sequences

.Here lub=glb = the constant term of the sequences

Monotonic sequences

In this section we intreducs an important class of sequences called monotonic sequences in which the terms are either continuously increasing or continiously decreasing.

Definition:

A sequences (a_n) is said to be monotonic increasing if $a_n \leq a_{n+1}$ for all n (a_n) is said to be monotonic decreasing if $a_n \geq a_{n+1}$ for all n $(a_n)$is said to be serictly monotonic increasing if $a_n < a_{n+1}$ for all n and strictly monotonic decreasing if $a_n > a_{n+1}$ for all n (a_n) is said to be monotonic if it is either monotonic increasing or monotonic decreasing.

Examples:

1)1,2,2,3,3,3,4,4,4,4,.......is a monotonic increasing sequence

2)1,2,3,4,.......n,..... is a strictly monotonic increasing sequence

3)1,1/2,1/3,1/4,.....1/n,......is a strictly monotonic decreasing sequence

4)The sequence(a_n)given 1,-1,1....is neither monotonic increasing nor decreasing.Hence (a_n) is not a monotonic sequence.

5) (2n-7/3n+2) is a monotonic incresing sequence

Proof:

a_n-a_{n+1}=2n-7/3n+2-2(n+1)-7/3(n+1)+2=-25/(3n+2)(3n+5)

$a_n < a_{n+1}$

Hence the sequence is monotonic increasing.

6)Consider the sequence (a_n)where (a_n)=1+1/1!+1/2!+....+1/n!   clearly (a_n) is a monotonic increasing sequence

Note:

A monotonic increasing sequence (a_n) is bounded below and a_1 is theg.l.b of the sequence.A monotonic decreasing sequence (a_n) is bounded above and a_1 is the l.u.b of the sequence.

Solved problems:

Problem 1: Show that if (a_n) is a monotonic sequence then($a_1+a_2+....a_n/n$) is also a monotonic sequence.

Solution:

Let (a_n) be a monotonic increasing sequence

$a_1 \leq a_2 \leq a_3 \leq .... \leq a_n \leq ....$

let $b_n=a_1+a_2+.....a_n/n$

Now $b_{n+1}-bn=a_1+....a_{n+1}/n+1-a_1+.....a_n/n$

$=na_{n+1}-(a_1+.....a_n)/n(n+1)$

$=na_{n+1}-(a_n+a_n+.....+a_n)/n(n+1)$

$=n(a_{n+1}-a_n)/n(n+1)$

≥ 0

$b_{n+1}\geq b_n$

b_n is monotonic increasing.

The proof is similon if (an)is monotonic decreasing.

Definition:

A sequence (a_n) is said to coverge to a number l if given there exists a positive integer m such that for all $n \geq m$ we say that l is the limit of the sequence and we write lim a_n=l or (a_n)=l

Note:1

(a_n) l if given there exists a natural number m such that for all $n \geq m$ all but a finite number of terms of the sequence liv within the interval

Note:2

The above definition does not give any method of finding the limit of a sequence in many cases by opsering the sequence carefully we can guess whether the limit exists or not and also the value of the limit.

Theorem3.1:

A sequence cannot converge to two different limits.

Proof:

Let (a_n) be a convergent sequence.

If passible let l_1 and l_2 be two distincl limits of (a_n)

Since(a_n) l ,there exists a natural number n_1 search that $(a_n-l_1)<1/2$ for all $n\geq n_1$

Since (a_n) l_2, there exists a natural n_2 such that

$(a_n-l_2)<1/2$ for all $n_1\geq n_2$

Let $m=\max\{n_1,n_2\}$

Then $(l_1-l_2)=(l_1-a_m+a_m-l_2)$

$\leq(a_m-l_1)+(a_m-l_2)$

clearly this is possible if and only if $l_1-l_2=0$.

Hence l_1-l_2.

CONVERGENT SEQUENCES

EXAMPLE :

= 0

Proof:

Let $\epsilon>0$ be given

Then $|1/n - 0| = 1/n < \epsilon$ if n $> 1/\epsilon$

Hence if we choose m to be any natural number such

That m $\geq 1/\epsilon$

Then $|1/n - 0| < \epsilon$ for all $n \geq m$.

Therefore = 0.

NOTE :

If ϵ=1/100, then m can be chosen to be any natural number greater than 100. In this example the choice of m depends on the given ϵ and [1/ϵ] + 1 is the smallest value of m that satisfies the requirements of the definition.

2. The constant sequence 1, 1, 1, ... converges to 1.

PROOF:

Let $\epsilon > 0$ be given

Let the given sequence be denoted by (a_n).

Then $a_n = 1$ for all n.

Therefore $| a_n - 1 | = | 1 - 1 | = 0 < \epsilon$ for all $n \in N$.

Therefore $| a_n - 1 | < \epsilon$ for all $n \geq m$ where m can be chosen to be any natural number.

depend on the given ϵ

3. $= 1$.

PROOF:

Let $\epsilon > 0$ be given.

Now, $\left| \frac{n+1}{n} - 1 \right| = \left| 1 + \frac{1}{n} - 1 \right| = |1/n|$.

Therefore if we choose m to be any natural number greater than $1/\epsilon$ we have, $|n+1/n - 1| < \epsilon$ for all $n \geq m$.

Therefore $\lim_{n\to\infty} \frac{n+1}{n} = 1$.

4. Lim $= 0$.

PROOF : Let $\epsilon > 0$ be given,

Then $|1/n^2 - 0| = 1/n^2 < 1/n$

Therefore $\left| \frac{1}{n^2} - 0 \right| < \epsilon$ for all $n \geq m$ where m is any natural number greater than $\frac{1}{\epsilon}$

Therefore $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

5. The sequence $((-1)^n)$ is not convergent.

Proof :

Suppose the sequence $((-1)^n)$ converges to l

Then given $\epsilon > 0$, there exists a natural number m such that $|(-1)^n - l| < \epsilon$ for all $n \geq m$.

$$|(-1)^n - (-1)^{n+1}| = |(-1)^n - l + l - (-1)^{n+1}|$$

$$\leq |(-1)^n - l| + |(-1)^{n+1} - l|$$

$$< \epsilon + \epsilon = 2\epsilon$$

But $|(-1)^n - (-1)^{n+1}| = 2$

Therefore $2 < 2\epsilon$ ie, $1 < \epsilon$ which is a contradiction since $\epsilon > 0$ is arbitrary.

The sequence $((-1)^n)$ is not convergent.

THEOREM 3.2

Any convergent sequence is a bounded sequence.

PROOF:

Let (a_n) be a convergent sequence.

Let $\lim_{n\to\infty} a_n = L$.

Let $\epsilon > 0$ be given. Then there exists $m \in N$ such that

$|a_n - l| < \epsilon$ for all $n \geq m$

Therefore $|a_n| < |l| + \epsilon$ for all $n \geq m$

Now let $k = \max\{|a_1|, |a_2|, \ldots, |a_{m-1}|,$

$|l| + \epsilon\}$

Then $|a_n| \leq k$ for all n

(a_n) is a bounded sequence.

NOTE:

The converse of the above theorem is not true. For example, the sequence $((-1)^n)$ is a bounded sequence. However it is not a convergent sequence.

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We now proceed to classify sequences which are not convergent as follows.

1. Sequences diverging to ∞

2.Sequences diverging to $-\infty$

3.Finetely oscillating sequences.

4.Infinetely oscillating sequences.

Difination

A sequence (a) is said to diverge to ∞ if given any real number $k>0$, there exists $m\in N$ such that $a_n>k$ for all $n\geq m$. In symbols we write $(a_n) \to \infty$ or $\lim_{n\to\infty} a_n=\infty$

Note. $(a_n) \to \infty$ or iff given any real number k>>0 there exists m€ N such that $a_n \in (k,\infty)$ for all $n\geq m$

Examples.

1.$(n) \to \infty$

Proof. Let $k > 0$ be any given real number.

Choose m to be any natural number such that $m>k$

Then $n>k$ for all $n\geq m$

$(n) \to \infty$.

2. $(n^2) \to \infty$.

Proof. Let $k > 0$ be any given real number.

Choose m to be any natural number such that $m > \sqrt{k}$.

Then $n^2 > k$ for all $n \geq m$

$(n^2) \to \infty$.

Proof. Let $k > 0$ be any given real number.

Then $2^n > k$ nlog 2 >log k.

$n > (\log k) \backslash \log 2$

Hence if we choose m to be any natural number such that $m > (\log k)/\log 2$, then $2^n > k$ for all $n \geq m$.

$(2^n) \to \infty$

Definition. A sequence (a^n) is said to diverge to s-∞ if given any real number $k < 0$ there exists

$m \in N$ such that $a_n < k$ for all $n \geq m$. In symbols we write $\lim_{n \to \infty} a_n = -\infty$.

Note. $(a_n) \to -\infty$ iff given any real number $k < 0$, there exists $m \in n$ such that $a_n \in (-\infty, k)$ for all $n \geq m$.

A sequence (a_n) is said to be divergent if either $(a_n) \to \infty$ or $(a_n) \to -\infty$.

Theorem 3.3 $(a_n) \to \infty$ iff $(-a_n) \to -\infty$.

Let $k < 0$ be any given real number . Since $(a_n) \to \infty$ there exists

$m \in N$ such that $a_n > -k$ for all $n \geq m$.

$-a_n < k$ for all $n \geq m$.

$(-a_n) \to -\infty$.

Similarly we can prove that if $(-a_n) \to -\infty$ then $(a_n) \to \infty$.

Examples. The sequences $(-n)$, $(-n^2)$ and (-2^n) diverge to $-\infty$.

Theorem 3.4 If $(a_n) \to \infty$ and a_n 0 for all $n \in N$ then $(1/a_n) \to 0$.

Proof . Let $\epsilon > 0$ be given. Since $(a_n) \to \infty$, there exists $m \in N$ Such that $a_n > 1/\epsilon$

For all $n \geq m$.

$1/a_n < \epsilon$ for all $n \geq m$.

$|1/a_n| <$ for all $n \geq m$.

$(1/a_n) \to 0$.

THEOREM 3.5

If $(a_n) \to 0$ and $a_n > 0$ for all $n \in N$, then $(\frac{1}{a_n}) \to \infty$

PROOF:

Let $K > 0$ *be* any given real number. Since there exists $m \in N$ such that $|a_n| < \frac{1}{k}$ for all $n \geq m$

$a_n < \frac{1}{k}$ for all $n \geq m$ (since $a_n > 0$)

$\frac{1}{a_n} > k$ for all $n \geq m$

$(\frac{1}{a_n}) \to \infty$

THEOREM 3.6

Any sequence (a_n) diverging to ∞ is bounded below but not bounded above.

PROOF :

Let $(a_n) \rightarrow \infty$. Then for any given real number $k > 0$ there exists $m \in N$ such that $(a_n) > k$ for all $n \geq m$. ……………(1)

K is not an upper bound of the sequence (a_n).

(a_n) is not bounded above.

Now let $l = \{a_1, a_2, \ldots\ldots\ldots a_m, k\}$.

From (1) we see that $a_n \geq l$ for all n.

(a_n) is bounded below.

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THEOREM 3.7

Any sequence (a_n) diverging to -∞ is bounded above but Not below.

PROOF :

Let (a_n)→-∞. Then for any given real numer k< 0 there exists m∈ N such that $a_n < K$ for all n≥ m. ....(1)

K is an upper bound of the sequence (a_n).

(a_n) is bounded above.

Now let l={a_1, a_2,............a_m,k}.

From (1) we see that $a_n \leq l$ for all n.

(a_n) is bounded above.

NOTE 1.

The converse of the above theorem is not true. For example,

The function, f:N→R defined by

f(n)={0 if n is odd

f(n)={$\frac{1}{2n}$ if n is even

determines the sequence 0,1,0,2,0,3,......which is bounded below and not bounded above.

Also for any real number k> 0,we cannot find a Natural number m such that $a_n > k$ for all n≥ m.

Hence this sequence does not diverge to ∞.

Similarly f:N→R given by

f(n)={0 if n is odd

f(n)={$-\frac{1}{2n}$ if n is even

determines the sequence 0,-1,0,-2,0,......... which is bounded above and not bounded below.

However this sequence does not diverge to $-\infty$.

NOTE 2.

By theorem 3.2 any convergent sequence is bounded. Hence by theorem 3.6 we see that any convergent sequence cannot diverge to ∞. Similarly by theorem 3.7 it cannot diverge to $-\infty$. Also any sequence diverging to ∞ cannot converge or diverge to $-\infty$ and any sequence diverging to $-\infty$ cannot Converge or diverge to ∞. Thus the three behaviours of a Sequence namely convergence, divergence to ∞ and divergence to $-\infty$ are mutually exclusive.

However these three types of behaviour of sequences are Not exhaustive since there exist sequence which neither converge nor diverge to ∞ nor diverge to $-\infty$.

Definition:

A sequence (a_n) which is neither convergent nor divergent to ∞ or $-\infty$ Is bounded is said to be an **oscillating sequence.** An oscillating sequence which is bounded is said to be **finitely oscillating.** An oscillating sequence which is unbounded is said to be infinitely **oscillating.**

Example:

1.consider the sequence $((-1)^n)$. *Since* this sequence is bounded it cannot diverge to ∞ or $-\infty$ (by theorem 3.6 and 3.7). Also this sequence is not convergent (by example 5 of 3.4). Hence $((-1)^n)$.is a finitely oscillating sequence.

2.The function f:N→R defined by

$f(n)=\{\frac{1}{2n}$ if n is even

$f(n)=\{\frac{1}{2(1-n)}$ if n is odd

determines the sequence 0,1,-1,2,-2,3,.......The range of this sequence is Z. Hence the sequence is neither bounded below nor bounded above.

Hence it cannot converge or diverge to $\pm\infty$.

This sequence is infinitely oscillating.

THE ALGEBRA LIMITS

In this section we prove a few simple theorems for sequences which are very useful in calculating

Limits of sequences.

Theorem 3.8

If $(a_n) \to a$ and $(b_n) \to b$ then $(a_n+b_n) \to a+b$

Proof:

Let $\varepsilon>0$ be given.

Now $| a_n+b_n -a-b | = | a_n-a+b_n-b |$

$\leq |a_n-a | + |b_n-b |$ ....(1)

Since $| a_n| \to a$ there exists a natural number n_1 such that,

$|a_n-a |< ½\varepsilon$ for all $n\geq n_1$ ....(2)

Since $|b_n | \to b$, there exists a natural number n_2 such that,

$|b_n-b |< ½\varepsilon$ for all $n\geq n_2$ ....(3)

Let $m=\max\{ n_1,n_2 \}$

Then $|a_n+b_n-a-b | < ½\varepsilon + ½\varepsilon = \varepsilon$ for all $n\geq m$

(by 1,2 and 3)

$\therefore (a_n+b_n) \to a+b$

Note: Similarly we can prove that

$(a_n-b_n) \to a-b$

Theorem 3.9

If $(a_n) \to a$ and $k \in R$ then $(Ka_n) \to ka$

Proof:

If $k=0$. (ka_n) is the constant sequence 0,0,0,.... and hence the result is trivial.

Now let $k=0$

Then $| ka_n-ka | = | k| \ |a_n-a |$ ....(1)

Let $\varepsilon>0$ be given.

Since $(a_n) \to a$, there exists $m \in N$ such that

$|a_n-a | < \varepsilon/|k|$ for all $n\geq m$ ....(2)

$\therefore | ka_n-ka |< \varepsilon$ for all $n\geq m$ (by 1 and 2)

$(ka_n) \to ka$

Theorem 3.10

If $(a_n)\to a$ and $(b_n)\to b$ then $(a_n b_n)\to ab$.

Proof:

Let $\varepsilon<0$ be given

Now, $|a_n b_n| = |a_n b_n - a_n b + a_n b - ab|$

$\leq |a_n b_n - a_n b| + |a_n b - ab|$

$= |a_n|\,|b_n - b| + |b|\,|a_n - a|$ ....(1)

Also, since $(a_n)\to a$, (a_n) is a bounded sequence. (by theorem 3.2)

$\therefore$ There exists a real number $k>0$ such that $|a_n|\leq K$ for all n ....(2)

Using (1) and (2) we get,

$|a_n b_n - ab| \leq k\,|b_n - b| + |b|\,|a_n - a|$ ....(3)

Now since $(a_n)\to a$ there exists a natural number n_1 such that

$|a_n - a| < \frac{\varepsilon}{2|b|}$ for all $n\geq n_1$ ....(4)

Since $(b_n)\to b$, there exists a natural number n_2 such that

$|b_n - b| < \frac{\varepsilon}{2k}$ for all $n\geq n_2$ ....(5)

Let $m= \max\{n_1, n_2\}$. Then $|a_n b_n - ab| < k\left(\frac{\varepsilon}{2k}\right) + |b|\left(\frac{\varepsilon}{2|b|}\right) = \varepsilon$ for all $n\geq m$.

(by 3,4 and 5)

Hence $(a_n b_n)\to ab$.

Theorem 3.11

If $(a_n)\to a$ and $a_n=0$ for all n and $a=0$, then $\left(\frac{1}{a_n}\right)\to\frac{1}{a}$.

Proof:

Let $\varepsilon>0$ be given. We have $\left|\frac{1}{a_n}-\frac{1}{a}\right| = \left|\frac{a_n - a}{a_n a}\right| = \frac{1}{|a_n||b_n|}\,|a_n - a|$ ....(1)

Now, $a\neq 0$. Hence $|a|>0$. Since $(a_n)\to a$ there exists $n_1 \epsilon N$ such that $|a_n - a|<½|a|$ for all $n\geq n_1$.

Hence $|a_n| >½\,|a|$ for all $n\geq n_1$ ....(2)

Using (1) and (2) we get, $\left|\frac{1}{a_n}-\frac{1}{a}\right| < \frac{2}{|a|^2}\,|a_n - a|$ for all $n\geq n_1$ ....(3)

Now since $(a_n)\to a$ there exists $n_2 \epsilon N$ such that $|a_n - a|<½\varepsilon|a|^2$ for all $n\geq n_2$ ....(4)

Let $m=\max\{n_1,n_2\}$.$\therefore \left|\frac{1}{a_n}-\frac{1}{a}\right| < \frac{2\varepsilon|a|^2}{|a|^2 2} = \varepsilon$ *for all* $n\geq m$ (by 3 and 4)

$\therefore \left(\frac{1}{a_n}\right) \to \frac{1}{a}$.

Corollary:

Let $(a_n)\rightarrow a$ and $(b_n)\rightarrow b$ when $b_n \neq 0$ for all n and $b \neq 0$.

Then $(\frac{a_n}{b_n}) \rightarrow \frac{a}{b}$

Proof:

$(\frac{1}{b_n}) \rightarrow \frac{1}{b}$ (by theorem 3.11)

$\therefore (\frac{a_n}{b_n}) \rightarrow \frac{a}{b}$ (by theorem 3.10)

Note:

Even if $\lim_{x\to\infty} a_n$ *and* $\lim_{x\to\infty} b_n$ do not exist, $\lim_{x\to\infty}(a_n + b_n)$ *and* $\lim_{x\to\infty} \frac{a_n}{b_n}$ may exist.
For example let $a_n = ((-1)^n)$ and $b_n = ((-1)^{n+1})$. Clearly $\lim_{x\to\infty} a_n$ *and* $\lim_{x\to\infty} b_n$ do not exists.

Now $(a_n + b_n)$ is the constant sequence 0,0,0,...... Each of $(a_n\, b_n)$ and (a_n/b_n) is the constant sequence -1,-1,-1,...... Hence $(a_n+b_n)\rightarrow 0$. $(a_n\, b_n)\rightarrow -1$ and $(a_n/b_n)\rightarrow -1$.

Theorem 3.12

If $(a_n) \to a$ then $(|a_n|) \to |a|$

Proof :

Let $\epsilon > 0$ be given

Now $||a_n| - |a|| \leq |a_n - a|$ -----------(1)

Since $(a_n) \to a$, there exists $m \in N$

Such that $|a_n| - |a| < \epsilon$ for all $n \geq m$.

Hence from (1) we get $||a_n| - |a|| < \epsilon$ for all $n \geq m$

Hence $(|a_n|) \to |a|$

Theorem 3.13

If $(a_n) \to a$ and $a_n \geq 0$ for all n then $a \geq 0$

Proof:

Suppose $a < 0$, then $-a > 0$

Choose ϵ such that $0 < \epsilon < -a$ so that $a + \epsilon < 0$

Now, since $(a_n) \to a$, there exist $m \in N$ such that $|a_n - a| < \epsilon$ for all $n \geq m$

$\therefore a - \epsilon < a_n < a + \epsilon$ for all $n \geq m$.

Now , since $a + \epsilon < 0$, we have $a_n < 0$

For all $n \geq m$ which is a contradiction since $a_n \geq 0$. Hence $a \geq 0$

Note:

In the above theorem if $a_n > 0$ for all n, we cannot say that $a > 0$. For example consider the sequence $(\frac{1}{n})$. Here $\frac{1}{n} > 0$ for all n and $(\frac{1}{n}) \to 0$

Theorem 3.14

If $(a_n) \to a$, $(b_n) \to b$ and $a_n \leq b_n$ for all n, then $a \leq b$.

Proof :

Since $a_n \leq b_n$ we have $b_n - a_n \geq 0$ for all n.

Also $(b_n - a_n) \to b - a$

$\therefore b - a \geq 0$

$\therefore a \leq b$

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Theorem 3.15

If $(a_n) \to l$, $(b_n) \to b$ and $a_n \leq c_n \leq b_n$ for all n, then $(c_n) \to l$

Proof :

Let $\in > 0$ be given

Since $(a_n) \to l$, there exists $n_1 \in N$

Such that $l - \in < a_n < l + \in$ for all $n \geq n_1$.

Similarly, there exists $n_2 \in N$.

Such that $l - \in < b_n < l + \in$ for all $n \geq m$

Let $m = \max\{n_1, n_2\}$

$\therefore l - \in < a_n \leq c_n \leq b_n < l + \in$ for all $n \geq m$

$l - \in < c_n < l + \in$ for all $n \geq m$

$|c_n - l| < \in$ for all $n \geq m$

$\therefore (c_n) \to l$

Theorem 3.16

If $(a_n) \to a$ and $a_n \geq 0$ for all n and $a \neq 0$ then $(\sqrt{an})$ - $\sqrt{a}$

Proof :

Since $a_n \geq 0$ for all n , $a \geq 0$

Now $|(\sqrt{an}) - \sqrt{a}| = \left|\frac{an - a}{\sqrt{an} + a}\right|$

Since, $(a_n) \to a \neq 0$

We obtain $a_n > \frac{1}{2} a$ for all $n \geq n_1$

$\sqrt{an} > \sqrt{(\frac{1}{2})a}$ for all $n \geq n_1$

$\therefore |(\sqrt{an}) - \sqrt{a}| < \frac{\sqrt{2}}{(\sqrt{2}+1)\sqrt{a}} |a_n - a|$ for all $n \geq n_1$ --------------------(1)

Now, let $\in > 0$ be given

Since $(a_n) \to a$, there exists $n_2 \in N$

Such that $|a_n - a| < \in \ (\sqrt{2} + 1)\sqrt{2}$ for all $n \geq n_2$ ----------------------(2)

Let $m = \max\{n_1, n_2\}$

Then $|a_n - a| < \in$ for all $n \geq m$ [by (1) and (2)]

$\therefore \sqrt{an} \to \sqrt{a}$

Theorem:3.17

If $(a_n) \to \infty$ and $(b_n) \to \infty$ then $(a_n + b_n) \to \infty$.

Proof:

Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $n_1 \in N$ such that $a_n > \frac{1}{2}k$ for all $n \geq n_1$.

Similarly, there exists $n_2 \in N$ Such that $b_n > \frac{1}{2}k$ for all $n \geq n_2$.

Let $m = \max\{n_1, n_2\}$

Then $a_n + b_n > k$ for all $n \geq m$.

$\therefore (a_n + b_n) \to \infty$

Theorem:3.18

If $(a_n) \to \infty$ and $(b_n) \to \infty$ and then $(a_n b_n) \to \infty$.

Proof:

Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $n_1 \in N$ such that $a_n > \sqrt{k}$ for all $n \geq n_1$.

Similarly, there exists $n_2 \in N$ Such that $b_n > \sqrt{k}$ for all $n \geq n_2$.

Let $m = \max\{n_1, n_2\}$

Then $a_n b_n > k$ for all $n \geq n_2$.

$\therefore (a_n b_n) \to \infty$.

Theorem:3.19

Let $(a_n) \to \infty$. Then

(i) if $c > 0$, $(c\, a_n) \to \infty$.

(ii) if $c < 0$, $(c\, a_n) \to -\infty$.

Proof:

(i) Let $c > 0$. Let $k > 0$ be any given real number.

Proof:

(i) Let $c > 0$. Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > k/c$ for all $n \geq m$.

$\therefore c\, a_n > k$ for all $n \geq m$.

$\therefore (c\, a_n) \to \infty$.

(ii) Let $c < 0$. Let $k < 0$ be any given real number.

There exists $m \in N$ such that $a_n > k/c$ for all $n \geq m$.

$\therefore c\,a_n < k$ for all $n \geq m$. (since $c < 0$)

$\therefore (c\,a_n) \to -\infty$.

Theorem:3.20

If $(a_n) \to \infty$ and (b_n) is bounded then $(a_n + b_n) \to \infty$.

Proof:

Since (b_n) is bounded, there exists a real number $m < 0$ such that

$b_n > m$ for all n. ....(1)

Now, Let $k > 0$ be any real number.

Since $m < 0$, $k - m > 0$.

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$m < 0$ such that

$b_n > m$ for all n. ....(1)

Now, Let $k > 0$ be any real number.

Since $m < 0$, $k - m > 0$.

Since $(a_n) \to \infty$, there exists $n_0 \in N$ such that

$A_n > k - m$ for all $n \geq n_0$ ....(2)

Add ①+②,

$a_n + b_n > k - m + m$

$a_n + b_n = k$ for all $n \geq n_0$.

$\therefore (a_n + b_n) \to \infty$.

Solved Problems

Problem:1

Show that $\lim\limits_{n\to\infty} \frac{3n^2+2n+5}{6n^2+4n+7} = \frac{1}{2}$

Solution:

$$a_n = \frac{3n^2+2n+5}{6n^2+4n+7} = \frac{n^2[3+\frac{2}{n}+\frac{5}{n^2}]}{n^2[6+\frac{4}{n}+\frac{7}{n^2}]}$$

$$a_n = \frac{3+\frac{2}{n}+\frac{5}{n^2}}{6+\frac{4}{n}+\frac{7}{n^2}}$$

Now, $\lim\limits_{n\to\infty}(3+\frac{2}{n}+\frac{5}{n^2}) = 3 + 2\lim\limits_{n\to\infty}\frac{1}{n} + 5\lim\limits_{n\to\infty}\frac{1}{n^2}$

$$=3+0+0$$

$$\lim_{n\to\infty}(3+\frac{2}{n}+\frac{5}{n^2}) = 3$$

Similarly ,

$$\lim_{n\to\infty}(6+\frac{4}{n}+\frac{7}{n^2}) = 6 + 4\lim_{n\to\infty}\frac{1}{n} + 7\lim_{n\to\infty}\frac{1}{n^2}$$

$$= 6+0+0$$

$$\lim_{n\to\infty}(6+\frac{4}{n}+\frac{7}{n^2}) = 6$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(3+\frac{2}{n}+\frac{5}{n^2})}{(6+\frac{4}{n}+\frac{7}{n^2})} = \frac{\lim\limits_{n\to\infty}(3+\frac{2}{n}+\frac{5}{n^2})}{\lim\limits_{n\to\infty}(6+\frac{4}{n}+\frac{7}{n^2})}$$

$$\lim_{n\to\infty} a_n = \frac{3}{6} = \frac{1}{2}$$

$$\boxed{\lim_{n\to\infty}\frac{3n^2+2n+5}{6n^2+4n+7} = \frac{1}{2}}$$

Hence proved .

Problem :2

Show that $\lim_{n\to\infty}\left(\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}\right)=\frac{1}{3}$

Solution :

We know that,

$$1^2+2^2+\cdots\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$\therefore \lim_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6n^3}$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{1}{6}\left[\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}=\frac{1}{6}\left[\left(1+\lim_{n\to\infty}\frac{1}{n}\right)\left(2+\lim_{n\to\infty}\frac{1}{n}\right)\right]$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}=\frac{1}{6}\left[(1+0)(2+0)\right]$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}=\frac{1}{6}\left[(1)\ (2)\right]$$

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}=\frac{2}{6}$$

$$\boxed{\lim_{n\to\infty}\frac{1^2+2^2+\cdots\cdots+n^2}{n^3}=\frac{1}{3}}$$

Hence proved .

Problem :3

Show that $\lim_{n\to\infty} \frac{n}{\sqrt{(n^2+1)}}$ =1

Solution:

$$\lim_{n\to\infty} \frac{n}{\sqrt{(n^2+1)}} = \lim_{n\to\infty} \frac{n}{n\sqrt{(1+\frac{1}{n^2})}}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{(n^2+1)}} = \lim_{n\to\infty} \frac{1}{\sqrt{(1+\frac{1}{n^2})}}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{(n^2+1)}} = \frac{1}{\lim_{n\to\infty} \frac{1}{\sqrt{(1+\frac{1}{n^2})}}}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{(n^2+1)}} = \frac{1}{\sqrt{\lim_{n\to\infty}(1+\frac{1}{n^2})}} \qquad (\therefore \lim_{n\to\infty} \frac{1}{n^2} = 0)$$

$$\boxed{\lim_{n\to\infty} \frac{n}{\sqrt{(n^2+1)}} = 1}$$

Hence proved .

Problem :4

Show that if $(a_n) \to 0$ *and* (b_n) is bounded then ,$(a_n b_n) \to 0$

Solution:

Since (b_n) is bounded , there exists K>0 .Such that $|b_n| \leq K$ for all n.

$$|a_n b_n| = |a_n|\,|b_n|$$

$$\therefore\ |a_n b_n| \leq K|a_n|$$

Now , let ε>0 be given .

Since $(a_n) \to 0$ there exists m∈ N . Such that $|a_n| < \frac{\varepsilon}{K}$ for all n $\geq m$

$$\therefore\ |a_n b_n| < \varepsilon \text{ for all } n \geq m$$

$$\therefore (a_n b_n) \to 0$$

Problem :5

Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$

Solution:

$|\sin n| \leq 1$ for all n.

$\therefore$ (sin n) is a bounded sequence .

Also $\left(\frac{1}{n}\right) \to 0$

We know that if $(a_n) \to 0$ *and* (b_n) is bounded then ,$(a_n b_n) \to 0$

$$\therefore \left(\frac{\sin n}{n}\right) \to 0$$

Problem 6:

Show that $\lim_{n\to\infty}(a^{1/n}) = 1$ where $a>0$ is any real number

Solution

Case 1: Let a=1. Then $a^{1/n}=1$ for each n

Hence $(a^{1/n})\to 1$

Case 2: Let a>1. Then $a^{1/n}>1$

$a^{1/n}=1+h_n$ where $h_n>0$

$a = (1+h_n)^n$

$=1+nh_n+\ldots\ldots+h_n^n$

$>1+nh_n$

$h_n<\frac{a-1}{n}$

$0 < h_n < \frac{a-1}{n}$

Hence $\lim_{n\to\infty} h_n=0$

$(a^{\frac{1}{n}}) = (1+h_n)\to 1$

Case 3: Let $0 < a < 1$. Then $\frac{1}{a}>1$

$(1/a)^{1/n} \to 1$

$(\frac{1}{a^{1/n}}) \to 1$

$(a^{1/n}) \to 1$

Problem 7:

Show that $\lim_{n\to\infty}(n^{1/n}) = 1$

Solution

Clearly $n^{\frac{1}{n}} \geq 1$ for all n

Let $n^{\frac{1}{n}} \geq 1+h_n$ Where $h_n \geq 0$

Then n= $(1+h_n)^n$

$=1+nh_n+nc_2 h_n^2+\ldots\ldots\ldots+h_n^n$

$>1/2\, n\,(n-1)\, h_n^2$

$h_n^2 < \frac{2}{n-1}$

$h_n < \sqrt{\frac{2}{n-1}}$

Since $\sqrt{\frac{2}{n-1}} \to 0$ and $h_n \geq 0$

$(h_n) \to 0$

$\lim_{n\to\infty} n^{\frac{1}{n}} = 1+\lim_{n\to\infty} h_n$

$(n^{\frac{1}{n}}) = (1+h_n) \to 1$

Problem 8:

Show that $\lim_{n\to\infty}(\frac{1}{\sqrt{(2n^2+1)}}+\frac{1}{\sqrt{2n^2+2}}+\ldots\ldots.+\frac{1}{\sqrt{2n^2+n}})=\frac{1}{\sqrt{2}}$

Solution

$$a_n = \frac{1}{\sqrt{2n^2+1}}+\frac{1}{\sqrt{2n^2+2}}+\ldots\ldots.+\frac{1}{\sqrt{2n^2+n}}$$

Then we have the inequality

$$\frac{n}{\sqrt{(2n^2+n)}} \leq a_n \leq \frac{n}{\sqrt{(2n^2+1)}}$$

$$\frac{1}{\sqrt{(2+1/n)}} \leq a_n \leq \frac{1}{\sqrt{(2+1/n^2)}}$$

Now $\lim_{n\to\infty}\frac{1}{\sqrt{(2+1/n)}} = \lim_{n\to\infty}\frac{1}{\sqrt{(2+1/n^2)}} = \frac{1}{\sqrt{2}}$

$$\lim_{n\to\infty} a_n = \frac{1}{\sqrt{2}}$$

Problem 9:

Given an example to show that if (a_n) is a sequence diverging to ∞ and (b_n) is a sequence diverging to -∞ then (a_n+b_n) need not be a divergent sequence

Solution

Let $(a_n) = n \ and \ (b_n)$ = (-n)

Clearly $(a_n)\rightarrow \infty$ and $(b_n)\rightarrow -\infty$

However (a_n+b_n) is the constant sequence 0, 0, 0…… converges to0